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A second kind integral equation formulation for the low Reynolds number interaction between a solid particle and a viscous drop

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Abstract. The low Reynolds number motion and deformation of a neutrally buoyant drop (immersed in a different viscous fluid) due to its interaction with a translating solid particle (immersed in the same fluid) is studied. This is achieved by means of a system of second-kind Fredholm integral equations. It is shown that the resulting system of integral equations possesses a unique continuous solution, and thus the proposed form of solution is assured to provide the unique regular solution of the present interaction problem.

1. Introduction

The practical importance of studying the motion of solid particles, drops and bubbles is due to their common occurrence in many industrial and biological systems, as well as in a number of technological processes. Chemical and metallurgical engineers rely on bubbles and drops for such operations as distillation, absorption, flotation, and spray drying. Mechanical engineers have studied droplet behaviour in connection with combustion operations, and bubbles in electromachining and boiling, etc. Particle suspensions play an important role in a wide variety of processes; flow of blood particles and proteins, pipeline transport of slurries, paper making, processing of ceramics and polymer or ceramic composite material, etc. The prediction of the structure and rheology of suspensions is thus of both of theoretical and practical interest.

One of the critical issues in the study of the suspension of such objects (solid particles, drops and bubbles) is the determination of their hydrodynamic interactions. Submerged objects moving through a viscous fluid interact quite strongly, with a persistence that decays as R^{-1} or R^{-2}, depending on whether the object exerts a net force on the fluid or not.

The investigation of the mechanics of solid particles, drops and bubbles has a long record in fluid dynamics, and continues to be a substantial portion of pure and applied research. A general review on the subject is given by Clift, Grace and Weber [1]. One of the most important problems within the general area of drop mechanics concerns the shape of these objects moving under the action of an external fluid. In this work we are interested in the motion and deformation of a neutrally buoyant drop (immersed in a different viscous fluid) due to its interaction with a translating solid particle (immersed in the same fluid). It is known that at small Reynolds number and constant external velocity, the flow associated with a single spherical drop, in an unbounded domain, satisfies all the necessary boundary conditions for steady motion independently of surface tension (for more details see Batchelor [2]). In our case, due to the shear on the drop surface induced by the exterior flow, the drop will deform and displace as the solid particle approaches it.

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Rallison and Acrivos [3] developed a numerical solution for the low Reynolds number deformation of a viscous drop suspended in a second fluid that is caused to shear, using the Green's integral representation formulae for the fluids inside and outside the drop. Integral representations analogous to those employed in potential theory exist for Stokes flows, and their use can be traced back to the work of Lorentz [4]; Odqvist [5] realized that the terms in the integral representation could be inspected separately and used these terms to create Stokes solutions. He called these terms the single and double layer.

The following integral representation formulae for the velocity fields are found from Green's formulae for the Stokes equation. In the unbounded domain Ω_e exterior to a single drop, with $\mathbf{u}^\infty$ as the asymptotic value at infinity,

$$u_i(\mathbf{x}) + \int_S K_{ij}(\mathbf{x},\mathbf{y})\,(u_j(\mathbf{y}))_e \mathrm{d}S_y = u_i^\infty(\mathbf{x}) + \int_S u_i^j(\mathbf{x},\mathbf{y})\,(\sigma_{jk}(\mathbf{u}(\mathbf{y})))_e\, n_k(\mathbf{y})\mathrm{d}S_y$$

for every $\mathbf{x} \in \Omega_e$, where $(u_i(\mathbf{y}))_e$ and $(\sigma_{ij}(\mathbf{u}(\mathbf{y})))_e$ are the values of the velocity field u_i and of the stress $\sigma_{ij}(\mathbf{u}(\mathbf{y}))$, respectively, at a point $\mathbf{y} \in S$ coming from Ω_e, and

$$u_i^j(\mathbf{x},\mathbf{y}) = -\frac{1}{8\pi}\left(\frac{\delta_{ij}}{r} + \frac{(x_i - y_i)(x_j - y_j)}{r^3}\right); \qquad r = |\mathbf{x} - \mathbf{y}|$$

is the fundamental solution of the Stokes equations, known as stokeslet at the point y, and

$$K_{ij}(\mathbf{x},\mathbf{y}) = \sigma_{ik}(\mathbf{u}^j)n_k = -\frac{3}{4\pi}\frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{r^5}\, n_k(\mathbf{y}).$$

In the bounded domain Ω_i interior to the drop, we have the following Green's representation formulae:

$$u_i(\mathbf{x}) - \int_S K_{ij}(\mathbf{x},\mathbf{y})\,(u_j(\mathbf{y}))_i \mathrm{d}S_y = -\frac{1}{\lambda}\int_S u_i^j(\mathbf{x},\mathbf{y})\,(\sigma_{jk}(\mathbf{u}(\mathbf{y})))_i\, n_k(\mathbf{y})\mathrm{d}S_y$$

for every $\mathbf{x} \in \Omega_i$, where $(u_i(\mathbf{y}))_i$ and $(\sigma_{ij}(\mathbf{u}(\mathbf{y})))_i$ are the values of the velocity field u_i and of the stress $\sigma_{ij}(\mathbf{u}(\mathbf{y}))$, respectively, at a point $\mathbf{y} \in S$ coming from Ω_i. In the above equations, the fluid velocity has been normalized using the viscosity μ of the carrying fluid, and the drop viscosity is $\lambda\mu$.

By requiring that the above flow fields satisfy the matching conditions at the drop interface, Rallison and Acrivos [3] found the following second-kind Fredholm integral equation for the unknown surface velocity $\mathbf{u}(x)$:

$$\frac{1}{2}u_i(\xi) + \frac{1-\lambda}{1+\lambda}\int_S K_{ij}(\boldsymbol{\xi},\mathbf{y})\, u_j(\mathbf{y})\, \mathrm{d}S_y = F_i(\boldsymbol{\xi}) \quad \text{for } \boldsymbol{\xi} \in S, \tag{1}$$

with

$$F_i(\boldsymbol{\xi}) = \frac{2}{1+\lambda}\left[u_i^\infty(\boldsymbol{\xi}) + \int_S u_i^j(\boldsymbol{\xi},\mathbf{y})\,\gamma\, n_j(y)\frac{\partial n_k}{\partial x_k}\, \mathrm{d}S_y\right].$$

The deformation of the surface interfaces is found from the kinematic conditions at such interfaces, which relate the rate of displacement of the interface to the normal velocity components at its surface.

When $\lambda = 1$, Eq. (1) takes the particularly simple form

$$u_i(\boldsymbol{\xi}) = u_i^\infty(\boldsymbol{\xi}) + \int_S u_i^j(\boldsymbol{\xi},\mathbf{y})\,\gamma\, n_j(\mathbf{y})\,\frac{\partial n_k}{\partial x_k}\, \mathrm{d}S_y,$$

which, in fact, is valid at all points $\mathbf{x}$, not only those on S.

It is known that the homogeneous form of Eq. (1) has only one eigensolution when $\lambda = 0$, corresponding to the case of a viscous drop of zero viscosity (gas bubble), and if $\lambda = \infty$, in the case of a viscous drop of infinite viscosity (solid particle), the six rigid–body motions for the drop are all eigensolutions (see Ladyzhenskaya [6]). Therefore, from Fredholm's alternative, it follows that the integral Eq. (1) does not admit a unique solution at these two poles of the resolvent. Also, it is clear that the resolvent does not have a pole at $\lambda = 1$ and, therefore, the same will be true in some small neighbourhood around $\lambda = 1$. Rallison and Acrivos [3] conjectured that, probably, there are no eigensolutions for $0 < \lambda < \infty$, since their numerical solution encountered no difficulties for values of λ tested in that range.

Power [7] proved, analytically, that the integral Eq. (1) possesses a unique continuous solution $\mathbf{u}(\mathbf{x})$ for any continuous datum $\mathbf{F}(\mathbf{x})$ when $0 < \lambda < \infty$; in other words, the resolvent of (1) does not have a pole in this range of λ. Therefore, the unique solution of integral Eq. (1) gives the drop-surface velocity for all possible cases, except when the object is a solid particle ($\lambda = \infty$) or a gas bubble ($\lambda = 0$).

In the case of a solid particle, ($\lambda = \infty$), Youngren and Acrivos [8] used the integral representation formulae for the exterior Stokes flow around the particle, to obtain a first kind Fredholm integral equation for the unknown surface traction. Youngren and Acrivos's first-kind approach has been used extensively in the literature for the numerical solution of different problems, including particle–particle interaction, the motion of a particle near a fluid interface or a rigid wall, the motion of particles in a container, etc. (for more details see Power and Wrobel [9]). As is known, Fredholm integral equations of the first kind generally give rise to unstable numerical schemes based upon discretization of the surface integrals involved, the instability manifesting itself in the ill–conditioning of the matrix approximation of the kernel. Nonetheless, it is possible to apply the discretization method if only low–order accuracy is desired and the system of linear equations to be solved is not too ill–conditioned, as appears to be the case in those works that use Youngren and Acrivos's method. On the other hand, solving an equation of the second kind is a well–posed problem.

Power and Miranda [10] explained how integral equations of the second kind can be obtained for general three–dimensional Stokes flows around a single particle. They observed that, although the double–layer representation which leads to a second-kind integral equation coming from the jump property of its velocity field across the density carrying surface can represent only those flow fields which correspond to surfaces that are force– and torque–free, the representation may be completed by adding terms that give arbitrary total force and torque in suitable linear combinations; to be more precise, a stokeslet and a rotlet located in the interior of the particle. The extension of Power and Miranda's method to multiple particles in an unbounded flow was given by Power [11] and Karrila *et al.* [12], and to a particle in a bounded flow by Power and Miranda [13] and Karrila and Kim [14]. In the last case, the fluid moves exterior to the particle surfaces but is contained by an exterior contour; therefore, it appears as an exterior flow when looking from the particle, but as an interior one from the exterior contour. The second kind formulation for this problem is not a trivial extension, since the completeness procedure of the deficient range of the double–layer potential due to the existence of an exterior container requires special care (for more details, see Kim and Karrila, [15]).

Karrila and Kim [14] and Karrila *et al.* [12] give an elegant mathematical interpretation of the above method. They observed that this method relates to Wielandt's deflation: by removing the end points of the spectrum of the integral operator of an integral equation of the second kind arising from a double–layer representation without any completion, those eigenvalues are moved without affecting the rest of the eigenvalues, providing a boundedly invertible operator and then allowing direct iterative solution. Karrila and Kim [14] call Power and Miranda's new method the *Completed Double Layer Boundary Integral Equation Method* (*CDL–IEM*), since it involves the idea of completing the deficient range of the double–layer operator. It is important to recognize here that Power and Miranda's completed method is an extension, to the Stokes system of equations, of Mikhlin's results [16] on the exterior Dirichlet problem for Laplace's equation.

A very important contribution of Kim and co–workers is that they show that the Completed Double Layer Boundary Integral Equation Method is the most efficient method to numerically solve the mobility problem, where the force and torque on each particle are specified and the unknown particle motion has to be determined.

Using an approach similar to Rallison and Acrivos [3] for a single drop, Manga and Stone [17] studied the interaction between two buoyancy-driven deformable drops, of viscosity $\lambda_1\mu$ and $\lambda_2\mu$. For that case they obtained a system of integral equations that can be written as the following single equation:

$$\frac{1}{2}\phi_i(\boldsymbol{\xi}) + \beta_1 \int_{S_1} K_{ij}(\boldsymbol{\xi},\mathbf{y})\ \phi_j(\mathbf{y})\ \mathrm{d}S_y + \beta_2 \int_{S_2} K_{ij}(\boldsymbol{\xi},\mathbf{y})\ \phi_j(y)\ \mathrm{d}S_y = f_i(\boldsymbol{\xi}), \tag{2}$$

where

$$\beta_1 = \frac{1-\lambda_1}{1+\lambda_1}, \qquad \beta_2 = \frac{1-\lambda_2}{1+\lambda_2},$$

$$f_i(\boldsymbol{\xi}) = \int_{S_1} u_i^j(\boldsymbol{\xi},\mathbf{y})\,\gamma_1\, n_j(\frac{\partial n_k}{\partial x_k}+\Delta\rho_1 g_k y_k)\ \mathrm{d}S_y + \int_{S_2} u_i^j(\boldsymbol{\xi},\mathbf{y})\,\gamma_2\, n_j(\frac{\partial n_k}{\partial x_k}+\Delta\rho_2 g_k y_k)\ \mathrm{d}S_y$$

and

$$\begin{aligned} \phi_i(\boldsymbol{\xi}) &= (1+\lambda_1)\, u_i^1(\boldsymbol{\xi}) \qquad &&\text{when} \quad \boldsymbol{\xi} \in S_1, \\ \phi_i(\boldsymbol{\xi}) &= (1+\lambda_2)\, u_i^2(\boldsymbol{\xi}) \qquad &&\text{when} \quad \boldsymbol{\xi} \in S_2. \end{aligned}$$

By considering the size of one of the drops to be infinite in the above formulation, Manga and Stone [18] studied the low Reynolds number buoyancy-driven translation of a deformable drop towards and through a fluid–fluid interface.

By reducing of the original system of integral equations to a single integral equation, Power [19] was able to conduct a theoretical analysis of the solubility of integral equation (2), by extending his previous work on a single drop. He showed that all poles of the resolvent of integral Eq. (2) are real, and that there is no pole when $0 > \lambda_1 > \infty$ and $0 > \lambda_2 > \infty$. Hence, by solving integral Eq. (2), we are able to study the interaction between two different viscous drops, but not the interaction between a solid particle or a gas bubble with a viscous drop.

This work presents a constructive way of finding the hydrodynamic interaction between a solid particle and a viscous drop. This is achieved by means of a system of second-kind Fredholm integral equations. The approach presented here is an extension of Power and Miranda's *Completed Double Layer Boundary Integral Equation Method.* It also shows that the resulting system of integral equations possesses a unique continuous solution, and thus the proposed form of solution is assured to provide the unique regular solution of the present interaction problem.

2. Statement of the problem

We will consider the low Reynolds number interaction between a translating solid particle in a viscous fluid, and a neutrally buoyant viscous drop, with viscosity ratio λ, which has interfacial surface tension γ. The governing equations for the fluid velocity $\mathbf{u}$ and pressure p are the Stokes equations, written here in the form:

$$\frac{\partial u_i}{\partial x_i} = 0; \qquad \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \tag{3}$$

where

$$\sigma_{ij} = \begin{cases} -p\delta_{ij} + (\partial u_i/\partial x_j + \partial u_j/\partial x_i) & \text{for } \mathbf{x} \in \Omega_e \\ -p\delta_{ij} + \lambda(\partial u_i/\partial x_j + \partial u_j/\partial x_i) & \text{for } \mathbf{x} \in \Omega_i \end{cases}. \tag{4}$$

Here, we have normalized the fluid velocity, using the viscosity μ of the carrying fluid; the drop viscosity is $\lambda\mu$. The flow fields have to satisfy the following asymptotic, boundary (at the particle surface S_1), and matching (at the drop surface S_2) conditions:

$$u_i \to 0, \quad p \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty, \tag{5}$$

$$u_i = U_i \quad \text{for every} \quad \boldsymbol{\xi} \in S_1 \tag{6}$$

and

$$[\mathbf{u}]_{S_2} = 0, \quad [\sigma_{ij} n_i]_{S_2} = \gamma n_j \frac{\partial n_k}{\partial x_k} \quad \text{for every} \quad \boldsymbol{\xi} \in S_2, \tag{7}$$

where $[\,]_{S_2}$ denotes the jump across the surface of the drop S_2 from the outside Ω_e (the carrying fluid domain) to the inside Ω_i (the viscous drop domain), $\mathbf{n}$ is the outward unit normal and $\partial n_j/\partial x_j$ is the surface curvature. It has been assumed that, during deformation, the drop remains smooth enough, so that the curvature can be approximated by the divergence of $\mathbf{n}$.

The rate of deformation of the drop surface is determined by the kinematic boundary condition, which states that the normal component of the fluid velocity at a point ξ of the surface drop is equal to the normal component of the surface velocity at that point:

$$\frac{d\xi_i}{dt} n_i = u_i n_i \quad \text{at} \quad \boldsymbol{\xi} \in S_2. \tag{8}$$

3. Uniqueness of solution

In order to facilitate our study of uniqueness of the problem stated in the previous section, we assume from the start that the flow field $(\mathbf{u}, p)$ is such that $\mathbf{u}(\mathbf{x})$ converges to zero like $|\mathbf{x}|^{-1}$, that its derivatives converge to zero like $|\mathbf{x}|^{-2}$, and that $p(\mathbf{x})$ converges to zero as $|\mathbf{x}|^{-2}$ as $|\mathbf{x}| \to \infty$. With the above assumptions concerning the behaviour of the carrying fluid at infinity, we obtain the following relations from Green's first identity for Stokes's equations:

$$\int_{S_2} \sigma_{ij}(\mathbf{u}, p)\, n_j\, u_i \,\mathrm{d}S = \frac{1}{2} \int_{\Omega_i} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right)^2 \mathrm{d}\Omega, \tag{9}$$

$$\int_{S_1} \sigma_{ij}(\mathbf{u}, p)\, n_j\, u_i \,\mathrm{d}S + \int_{S_2} \sigma_{ij}(\mathbf{u}, p)\, n_j\, u_i \,\mathrm{d}S = -\frac{1}{2} \int_{\Omega_e} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right)^2 \mathrm{d}\Omega. \tag{10}$$

Taking the difference between the above two equations, and substituting the boundary and matching conditions at the particle and drop surfaces, respectively, we obtain:

$$\int_{S_1} \sigma_{ij}(\mathbf{u},p)\, n_j\, U_i \,\mathrm{d}S + \int_{S_2} \gamma n_j \frac{\partial n_k}{\partial x_k}\, u_i \,\mathrm{d}S =$$

$$-\frac{1}{2}\left\{ \int_{\Omega_e} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)^2 \mathrm{d}\Omega + \int_{\Omega_i} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)^2 \mathrm{d}\Omega \right\}. \tag{11}$$

It is clear from the above equation that the present problem does not have more than one solution. Indeed, if the flow field satisfies an homogeneous boundary condition on $\boldsymbol{\xi} \in S_1 \cup S_2$, *i.e.* $\mathbf{U} = 0$ and $\gamma = 0$, then (11) implies

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = 0 \quad \text{for } x \in \Omega_e \cup \bar{\Omega}_i, \tag{12}$$

a system that is known to have six linearly independent solutions, corresponding to the motion of the fluid as a rigid body. Therefore, $(\mathbf{u}, p)$ vanishes throughout space, since such a null field is the only rigid–motion continuous velocity compatible with the asymptotic behaviour of the flow field at infinity.

4. Second kind integral equation solution

Following Power and Miranda's completed method, we will seek the solution of the above interaction problem in the following form. In the unbounded domain exterior to the particle and the drop, we will represent the velocity field as:

$$u_i(\mathbf{x}) = \int_{S_1} K_{ij}(\mathbf{x},\mathbf{y})\phi_j(\mathbf{y})\,\mathrm{d}S_y - \int_{S_2} u_i^j(\mathbf{x},\mathbf{y})\phi_j(\mathbf{y})\,\mathrm{d}S_y + u_i^j(\mathbf{x})\alpha_j + r_i^j(\mathbf{x})w_j, \tag{13}$$

together with its corresponding pressure

$$p(\mathbf{x}) = \frac{1}{2\pi}\frac{\partial}{\partial x_j}\int_{S_1} \frac{x_k - y_k}{r^3}\phi_k(\mathbf{y})n_j(\mathbf{y})\,\mathrm{d}S_y + \frac{1}{4\pi}\int_{S_2} \frac{x_k - y_k}{r^3}\phi_k(\mathbf{y})\,\mathrm{d}S_y - \frac{1}{4\pi}\frac{x_j}{R^3}\alpha_j \tag{14}$$

for every $\mathbf{x} \in \Omega_e$. Here $R = |\mathbf{x}|$ is the distance from the origin, chosen at the centre of the solid particle, to a point x in the flow field; and in the bounded domain interior to the drop we have

$$u_i(\mathbf{x}) = \int_{S_1} K_{ij}(\mathbf{x},\mathbf{y})\phi_j(\mathbf{y})\,\mathrm{d}S_y - \int_{S_2} u_i^j(\mathbf{x},\mathbf{y})\phi_j(\mathbf{y})\,\mathrm{d}S_y + u_i^j(\mathbf{x})\alpha_j + r_i^j(\mathbf{x})w_j \tag{15}$$

with

$$p(\mathbf{x}) = \frac{\lambda}{2\pi}\frac{\partial}{\partial x_j}\int_{S_1} \frac{x_k - y_k}{r^3}\phi_k(\mathbf{y})n_j(\mathbf{y})\,\mathrm{d}S_y + \frac{\lambda}{4\pi}\int_{S_2} \frac{x_k - y_k}{r^3}\phi_k(\mathbf{y})\,\mathrm{d}S_y - \frac{\lambda}{4\pi}\frac{x_j}{R^3}\alpha_j \tag{16}$$

for every $\mathbf{x} \in \Omega_i$.

The function

$$r_i^j(\mathbf{x}) = \frac{1}{8\pi}\frac{\varepsilon_{ilk}\delta_{lj}x_k}{R^3} \tag{17}$$

is a singular solution of the non–homogeneous Stokes system of equations, with non–homogeneous term (forcing function) given by $\varepsilon_{ilk}(\partial/\partial x_k)\,\delta_{lj}\delta(\mathbf{x})$, called rotlet.

It can be observed that in the above representational formulae, to the combination of a double–layer potential on the surface S_1:

$$W_i(\mathbf{x}) = \int_{S_1} K_{ij}(\mathbf{x},\mathbf{y})\phi_j(\mathbf{y})\ \mathrm{d}S_y \tag{18}$$

and a single layer potential on the surface S_2:

$$V_i(\mathbf{x}) = -\int_{S_2} u_i^k(\mathbf{x},\mathbf{y})\phi_k(\mathbf{y})\ \mathrm{d}S_y, \tag{19}$$

we have added a stokeslet $\mathbf{u}^j(\mathbf{x})$ located at the origin, whose strength is given by the constant vector $\boldsymbol{\alpha}$, and a rotlet $\mathbf{r}^j(\mathbf{x})$, also located at the origin, with strength equal to the constant vector $\mathbf{w}$.

This representation automatically satisfies the governing equations for the flow field in the exterior and interior domains, and the velocity matching condition at the drop surface S_2, due to the continuity property of the single-layer potential.

It is convenient, for later use, to choose $\boldsymbol{\alpha}$ and $\mathbf{w}$ as follows:

$$\alpha_i = \int_{S_1} \phi_j(\boldsymbol{\xi})\varphi_j^i(\boldsymbol{\xi})\ \mathrm{d}S_\xi \qquad \text{for} \quad i = 1,2,3, \tag{20}$$

$$w_i = \int_{S_1} \phi_j(\boldsymbol{\xi})\varphi_j^{i+3}(\boldsymbol{\xi})\ \mathrm{d}S_\xi \qquad \text{for} \quad i = 1,2,3, \tag{21}$$

where φ^i for $i = 1,2,\cdots 6$ give the motion of the fluid as a rigid body, taken to be:

$$\begin{aligned} \varphi^i(\mathbf{x}) &= (\delta_{1i},\delta_{2i},\delta_{3i}) \qquad \text{for} \quad i = 1,2,3 \\ \varphi^4(\mathbf{x}) &= (0,x_3,-x_2), \quad \varphi^5(\mathbf{x}) = (-x_3,0,x_1), \quad \varphi^6(\mathbf{x}) = (x_2,-x_1,0). \end{aligned} \tag{22}$$

As is well known, a stokeslet located at the origin exerts a total force equal to its strength and zero total torque on any closed surface enclosing it, and a rotlet exerts a total torque equal to its strength and zero total force on any surface enclosing it. Therefore, since a hydrodynamic double–layer potential with density carrying surface S_1 yields zero total force and torque on S_1, and a single-layer potential with a density carrying surface S_2 exerts a total force and torque on S_2 equal to

$$-\int_{S_2} \phi_i(\mathbf{y})\varphi_i^k\ \mathrm{d}S_y$$

for $k = 1,2,3$ and $k = 4,5,6$, respectively, it can be concluded that the total force and torque on the surface S_1 resulting from the flow field defined by equations (13)–(16) will be equal to $\boldsymbol{\alpha}$ and $\mathbf{w}$, respectively, and the total force and torque on the surface S_2 will be

$$F_i = -\int_{S_2} \phi_i(\mathbf{y})\varphi_i^k\ \mathrm{d}S_y \qquad \text{for} \quad k = 1,2,3,$$

$$T_i = -\int_{S_2} \phi_i(\mathbf{y})\varphi_i^k\ \mathrm{d}S_y \qquad \text{for} \quad k = 4,5,6.$$

For more details about the dynamic properties of the singularities considered here, as well as those of the double–layer and single–layer potentials, see Power and Wrobel [9].

Letting a point $\mathbf{x} \in \Omega_e$ approach a point $\boldsymbol{\xi} \in S_1$, we obtain from the above representational formulae for the exterior velocity field

$$U_i(\boldsymbol{\xi}) = -\frac{1}{2}\phi_i(\boldsymbol{\xi}) + \int_{S_1} K_{ij}(\boldsymbol{\xi},\mathbf{y})\phi_j(\mathbf{y})\,\mathrm{d}S_y - \int_{S_2} u_i^k(\boldsymbol{\xi},\mathbf{y})\phi_k(\mathbf{y})\,\mathrm{d}S_y + u_i^j(\boldsymbol{\xi})\alpha_j + r_i^j(\boldsymbol{\xi})w_j \qquad \boldsymbol{\xi} \in S_1, \tag{23}$$

where use has been made of the discontinuity property of a double-layer potential across the density-carrying surface, and the boundary condition at the surface of the solid particle. Henceforth, we assume that $\mathbf{U}$ is continuous on S_1.

Taking the surface tension of the flow field exterior to the particle and the drop, when $\mathbf{x} \in \Omega_e$ tends to a point $\boldsymbol{\xi} \in S_2$, we have:

$$\sigma_{ij}(\mathbf{u})_e n_j = -\frac{1}{2}\phi_i(\boldsymbol{\xi}) - \int_{S_2} K_{ji}(\mathbf{y},\boldsymbol{\xi})\phi_j(\mathbf{y})\,\mathrm{d}S_y + \sigma_{ik}\Big(\int_{S_1} K_{ij}(\boldsymbol{\xi},\mathbf{y})\phi_k(\mathbf{y})\,\mathrm{d}S_y\Big)n_k + K_{ij}(\boldsymbol{\xi})\alpha_j + T_{ij}(\boldsymbol{\xi})w_j, \qquad \xi \in S_2, \tag{24}$$

where

$$T_{ij}(\boldsymbol{\xi}) = \sigma_{ik}(\mathbf{r}^k(\boldsymbol{\xi}))n_j$$

is the surface tension due to a rotlet.

Similarly, taking the surface tension of the flow field interior to the drop, when $x \in \Omega_i$ tends to a point $\xi \in S_2$, we have:

$$\sigma_{ij}(\mathbf{u})_i n_j = \lambda\Big\{\frac{1}{2}\phi_i(\boldsymbol{\xi}) - \int_{S_2} K_{ji}(\mathbf{y},\boldsymbol{\xi})\phi_j(\mathbf{y})\,\mathrm{d}S_y + \sigma_{ik}\Big(\int_{S_1} K_{ij}(\boldsymbol{\xi},\mathbf{y})\phi_k(\mathbf{y})\,\mathrm{d}S_y\Big)n_k + K_{ij}(\boldsymbol{\xi})\alpha_j + T_{ij}(\boldsymbol{\xi})w_j\Big\}, \qquad \boldsymbol{\xi} \in S_2, \tag{25}$$

where use has been made of the discontinuity property of the surface tension of a single-layer potential across the density carrying surface. Here, $\sigma_{ij}(\mathbf{u})_e$ is the limiting value of the stress $\sigma_{ij}(\mathbf{u})$ when a point x tends to a point $\boldsymbol{\xi} \in S_2$ coming from Ω_e, and $\sigma_{ij}(\mathbf{u})_i$ is the limiting value of the stress when x tends to $\boldsymbol{\xi} \in S_2$ coming from Ω_i.

Substituting (24) and (25) in the surface-traction matching condition at the drop surface, we obtain

$$\gamma n_j\frac{\partial n_k}{\partial x_k} = (1-\lambda)\Big\{-\int_{S_2} K_{ji}(\mathbf{y},\boldsymbol{\xi})\phi_j(\mathbf{y})\,\mathrm{d}S_y + \sigma_{ik}\Big(\int_{S_1} K_{ij}(\boldsymbol{\xi},\mathbf{y})\phi_k(\mathbf{y})\,\mathrm{d}S_y\Big)n_k + K_{ij}(\boldsymbol{\xi})\alpha_j + T_{ij}(\boldsymbol{\xi})w_j\Big\} - \frac{(1+\lambda)}{2}\phi_i(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in S_2. \tag{26}$$

Eqs. (23) and (26) give a system of Fredholm second-kind integral equations for the unknown vector density ϕ at the surfaces S_1 and S_2. An apparent difficulty in the numerical solution of the above system of integral equations is that the integrals appear to be singular of the Cauchy type. In fact, they are weakly singular, behaving as $r^{-2+\alpha}$ when r tends to zero, since

$$K_{ij}(\mathbf{x},\mathbf{y}) = -\frac{3}{4\pi}\frac{(x_i-y_i)(x_j-y_j)(x_k-y_k)}{r^5}n_k(\mathbf{y}) = \frac{3}{4\pi r^2}\frac{\partial r}{\partial \xi_i}\frac{\partial r}{\partial \xi_j}\frac{\partial r}{\partial n_y} \tag{27}$$

and

$$\frac{\partial r}{\partial n_y} < Er^{\alpha} \tag{28}$$

with α the Lyapunov exponent ($0 < \alpha \leq 1$).

When $\lambda = 1$, Eq. (26) takes the particularly simple form

$$\gamma n_j \frac{\partial n_k}{\partial x_k} = -\phi_i(\boldsymbol{\xi}), \qquad \boldsymbol{\xi} \in S_2$$

and hence, Eq. (23) reduces to

$$U_i(\xi) - \gamma\int_{S_2} u_i^k(\xi, y)n_k(y)\frac{\partial n_l}{\partial y_l}\,\mathrm{d}S_y =$$
$$-\frac{1}{2}\phi_i(\boldsymbol{\xi}) + \int_{S_1} K_{ij}(\boldsymbol{\xi},\mathbf{y})\phi_j(\mathbf{y})\,\mathrm{d}S_y + u_i^j(\boldsymbol{\xi})\alpha_j + r_i^j(\boldsymbol{\xi})w_j, \qquad \boldsymbol{\xi} \in S_1,$$

which possesses a unique continuous solution ϕ in accordance with the completed double-layer theory. Therefore, it is clear that the resolvent of the system of integral equations (23) and (26) does not have a pole at $\lambda = 1$ and consequently, the same will be true in a small neighbourhood around $\lambda = 1$.

Generally speaking, integral equations (23) and (26) cannot be solved in closed form, and the solution has to be found by numerical approximation methods. However, it is known that such methods can only be applied with confidence when the solubility of the integral equation has been established beforehand; moreover, a question concerns the solubility with an arbitrary right–hand member for a given value of the parameter λ.

In order to show that the above system of integral equations possesses a unique continuous solution ϕ for a continuous datum $\mathbf{U}$ and $\gamma\mathbf{n}\partial n_k/\partial x_k$, it is sufficient, according to Fredholm's alternative, to show that the following homogeneous system for ϕ^0 admits only the trivial solution in the space of continuous functions:

$$-\frac{1}{2}\phi_i^0(\boldsymbol{\xi}) + \int_{S_1} K_{ij}(\boldsymbol{\xi},\mathbf{y})\phi_j^0(\mathbf{y})\,\mathrm{d}S_y$$
$$-\int_{S_2} u_i^k(\boldsymbol{\xi},\mathbf{y})\phi_k^0(\mathbf{y})\,\mathrm{d}S_y + u_i^j(\boldsymbol{\xi})\alpha_j^0 + r_i^j(\boldsymbol{\xi})w_j^0 = 0, \qquad \boldsymbol{\xi} \in S_1 \tag{29}$$

and

$$\frac{1-\lambda}{1+\lambda}\Big\{-\int_{S_2} K_{ji}(\mathbf{y},\boldsymbol{\xi})\phi_j^0(\mathbf{y})\,\mathrm{d}S_y + \sigma_{ik}\Big(\int_{S_1} K_{ij}(\boldsymbol{\xi},\mathbf{y})\phi_k^0(\mathbf{y})\,\mathrm{d}S_y\Big)n_k$$
$$+ K_{ij}(\boldsymbol{\xi})\alpha_j^0 + T_{ij}(\boldsymbol{\xi})w_j^0\Big\} - \frac{1}{2}\phi_i^0(\boldsymbol{\xi}) = 0, \qquad \boldsymbol{\xi} \in S_2, \tag{30}$$

where

$$\alpha_i^0 = \int_{S_1} \phi_j^0(\boldsymbol{\xi})\varphi_j^i(\boldsymbol{\xi})\,\mathrm{d}S_\xi \qquad \text{for} \quad i = 1, 2, 3, \tag{31}$$

$$w_i^0 = \int_{S_1} \phi_j^0(\boldsymbol{\xi})\varphi_j^{i+3}(\boldsymbol{\xi})\ \mathrm{d}S_\xi \qquad \text{for} \quad i = 1,2,3. \tag{32}$$

From the uniqueness of the solution of the present problem, and the system of homogeneous integral equations (29) and (30), it follows that the pair of vector fields $\mathbf{V}^1$, $\mathbf{V}^2$ defined below are equal in $\Omega_e \cup \bar{\Omega}_i$:

$$V_i^1(\mathbf{x}) = \int_{S_1} K_{ij}(\mathbf{x},\mathbf{y})\phi_j^0(\mathbf{y})\ \mathrm{d}S_y + r_i^j(\mathbf{x})w_j^0, \tag{33}$$

$$V_i^2(\mathbf{x}) = \int_{S_2} u_i^j(\mathbf{x},\mathbf{y})\phi_j^0(\mathbf{y})\ \mathrm{d}S_y - u_i^j(\mathbf{x})\alpha_j^0. \tag{34}$$

On the other hand, since $\mathbf{V}^1$ yields zero total force on the particle surface S_1, and $\mathbf{V}^2$ yields a non–zero total force there, it follows that the resulting force on S_1 exerted by $\mathbf{V}^2$ must be zero. Then

$$\alpha_i^0 = \int_{S_1} \phi_j^0(\boldsymbol{\xi})\varphi_j^i(\boldsymbol{\xi})\ \mathrm{d}S_\xi = 0 \qquad \text{for} \quad i = 1,2,3. \tag{35}$$

In a similar way, the resulting torque exerted by $\mathbf{V}^1$ on the surface S_1 is equal to $\mathbf{w}^0$, and the torque due to $\mathbf{V}^2$ is equal to zero. Then it follows from the above flow field identity that

$$w_i^0 = \int_{S} \phi_j^0(\boldsymbol{\xi})\varphi_j^{i+3}(\boldsymbol{\xi})\ \mathrm{d}S_\xi = 0 \qquad \text{for} \quad i = 1,2,3. \tag{36}$$

Therefore, Eqs. (33)–(36) imply that the double-layer potential with density carrying surface S_1, has to be equal to minus the single-layer potential with density carrying surface S_2 in $\Omega_e \cup \bar{\Omega}_i$, *i.e.*

$$\int_{S_1} K_{ij}(\mathbf{x},\mathbf{y})\phi_j^0(\mathbf{y})\ \mathrm{d}S_y = \int_{S_2} u_i^j(\mathbf{x},\mathbf{y})\phi_j^0(\mathbf{y})\ \mathrm{d}S_y. \tag{37}$$

In particular, it follows that the surface traction of the double-layer potential at points on the surface S_2 is given by

$$\sigma_{ik}\left(\int_{S_1} K_{ij}(\boldsymbol{\xi},\mathbf{y})\phi_k^0(\mathbf{y})\ \mathrm{d}S_y\right)n_k = \frac{1}{2}\phi_i^0(\boldsymbol{\xi}) + \int_{S_2} K_{ji}(\mathbf{y},\boldsymbol{\xi})\phi_j^0(\mathbf{y})\ \mathrm{d}S_y \tag{38}$$

coming from Ω_e, and

$$\sigma_{ik}\left(\int_{S_1} K_{ij}(\boldsymbol{\xi},\mathbf{y})\phi_k^0(\mathbf{y})\ \mathrm{d}S_y\right)n_k = -\frac{1}{2}\phi_i^0(\boldsymbol{\xi}) + \int_{S_2} K_{ji}(\mathbf{y},\boldsymbol{\xi})\phi_j^0(\mathbf{y})\ \mathrm{d}S_y \tag{39}$$

coming from Ω_i, which are identical owing to the regular behaviour of the double-layer potential at any point exterior to its density-carrying surface.

Substituting (35), (36) and (38) in (30), we get:

$$\frac{2\lambda}{1+\lambda}\phi_i^0 = 0, \quad \text{for every} \quad \boldsymbol{\xi} \in S_2.$$

From the above expression it follows that $\phi^0 = 0$ on S_2 as long as $\lambda \neq 0$, the case for $\lambda = 0$ corresponds to a viscous drop of zero viscosity (gas bubble). In a similar way, substituting (35), (36) and (39) in (30), we get:

$$\frac{2}{1+\lambda}\phi_i^0 = 0, \quad \text{for every} \quad \boldsymbol{\xi} \in S_2,$$

implying that $\phi^0 = 0$ on S_2 as long as $\lambda < \infty$, the case for $\lambda = \infty$ corresponds to a viscous drop of infinite viscosity (solid particle).

Substituting the null value of the single-layer density and the relations (35), (36) for the zero strength of the stokeslet and rotlet, we observe that Eq. (29) reduces to:

$$-\frac{1}{2}\phi_i^0(\boldsymbol{\xi}) + \int_{S_1} K_{ij}(\boldsymbol{\xi},\mathbf{y})\phi_j^0(\mathbf{y})\ \mathrm{d}S_y = 0 \qquad \text{for every} \quad \boldsymbol{\xi} \in S_1. \tag{40}$$

The above homogeneous equation has precisely six linearly independent solutions φ^k, for $k = 1, 2, \cdots, 6$, defined by the previously given rigid-body motion vectors. Then, necessarily $\phi_i^0 = \sum_{k=1}^{6} C_k \varphi_i^k$ for $i = 1, 2, 3$, on S_1, where $C_1, C_2, \cdots, C_6$ are some real constants.

The above general solution for ϕ^0, and Eqs. (35) and (36) imply, for $i = 1, 2, 3$, that:

$$C_k \int_{S_1} \varphi_j^k(\mathbf{y})\varphi_j^i(\mathbf{y})\ \mathrm{d}S_y = 0, \tag{41}$$

$$C_k \int_{S_1} \varphi_j^k(\mathbf{y})\varphi_j^{i+3}(\mathbf{y})\ \mathrm{d}S_y = 0. \tag{42}$$

The above linear algebraic system for $C_1, C_2, \cdots C_6$ only admits the trivial solution, implying that $\phi_i^0 = 0$ on S_1, because the determinant of (41)–(42) has the term

$$\int_{S_1} \varphi_j^l(\mathbf{y})\varphi_j^q(\mathbf{y})\ \mathrm{d}S_y \tag{43}$$

as element in its l-th row and q-th column, and is thus the Gram determinant for the vector functions $\varphi^1, \varphi^2, \cdots \varphi^6$ with a non–vanishing value, on account of the linear independence of $\varphi^k, k = 1, 2, \cdots 6$.

Therefore, for $0 < \lambda < \infty$ the homogeneous system of integral equations (29) and (30) admits only the trivial solution, or equivalently the system (23) and (26) possesses a unique continuous solution on such a range of λ, and then the Stokes flow field defined by this ϕ, using Eqs. (13)–(16) provides the solution of the present interaction problem.

After obtaining the drop surface velocity by substituting the unique vector density solution of (23) and (26) into the flow field (13)–(16), we determine the drop deformation by means of the kinematic condition.

From the stated dynamic properties of the singularities considered here, as well as those of the double–layer and single layer potentials, it is found that the total force and torque exerted on S_1 by the flow field are:

$$F_i = \alpha_i = \textstyle\int_{S_1} \phi_j(\boldsymbol{\xi})\varphi_j^i(\boldsymbol{\xi})\ \mathrm{d}S_\xi \qquad \text{for} \quad i = 1, 2, 3, \tag{45}$$

$$T_i = w_i = \textstyle\int_{S_1} \phi_j(\boldsymbol{\xi})\varphi_j^{i+3}(\boldsymbol{\xi})\ \mathrm{d}S_\xi \qquad \text{for} \quad i = 1, 2, 3, \tag{46}$$

and those on the surface S_2 are:

$$F_i = -\textstyle\int_{S_2} \phi_j(\boldsymbol{\xi})\varphi_j^i(\boldsymbol{\xi})\ \mathrm{d}S_\xi \qquad \text{for} \quad i = 1, 2, 3, \tag{47}$$

$$T_i = -\textstyle\int_{S_2} \phi_j(\boldsymbol{\xi})\varphi_j^{i+3}(\boldsymbol{\xi})\ \mathrm{d}S_\xi \qquad \text{for} \quad i = 1, 2, 3. \tag{49}$$

It can be observed that, when an initial spherical drop is far apart from the solid particle, in such a way that the single-layer density solution of (26) is only proportional to the normal vector, at its density carrying surface, then Eq. (23) reduces to that corresponding to a single solid particle given by Power and Miranda [10], due to the fact that a single layer with the normal vector as a density yields zero velocity field in the entire space. As the solid particle approaches the drop, the single layer density in (26) diverges from the normal vector yielding a non–zero velocity field, which finally will affect the dynamics of both the solid and the drop.

5. Conclusions

A well-posed system of Fredholm integral equations of the second kind for the problem of the low-Reynolds-number interaction between a neutrally buoyant viscous drop and a translating solid particle (immersed in the same fluid) has been found. We achieved this by completing the deficient range of the system of integral equations obtained when the flow field is represented by a linear combination of a single-layer potential, with the drop surface as the density carrying surface, and a double-layer potential, with the particle surface as the density-carrying surface (completed method).

The system of integral equations obtained in this work can be used as a basis for the numerical solution of the present interaction problem, using standard boundary element techniques, leading to an efficient and stable numerical scheme for problems of arbitrary geometries. This is because it is known that such a method can be applied with confidence only in the case when the solvability of the integral equation has been established beforehand.

One of the advantages of the present method is that the evaluation of the total force and torque upon the solid particle is found directly from the expressions for the strength of the corresponding stokeslet and rotlet, respectively, used for completing the deficient range of the original system of integral equations.

Since the completed method is amenable to iterative solution, it is expected that the same will be true for the resulting linear algebraic system of equations, found after discretization and numerical integration of the obtained system of integral equations, allowing the technique to be used in the case of large-scale problems.

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